Stochastic relaxational dynamics applied to finance: Towards non-equilibrium option pricing theory

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Abstract. Non-equilibrium phenomena occur not only in the physical world, but also in finance. In this work, stochastic relaxational dynamics (together with path integrals) is applied to option pricing theory. Equilibrium in financial markets is defined as the absence of arbitrage, *i.e.* profits "for nothing". A recently proposed model (by Ilinski *et al.*) considers fluctuations around this equilibrium state by introducing a relaxational dynamics with random noise for intermediate deviations called "virtual" arbitrage returns. In this work, the model is incorporated within a martingale pricing method for derivatives on securities (*e.g.* stocks) in incomplete markets using a mapping to option pricing theory with stochastic interest rates. The arbitrage return is considered as a component of a fictitious short-term interest rate in a virtual world. The influence of intermediate arbitrage returns on the price of derivatives in the real world can be recovered by performing an average over the (non-observable) arbitrage return at the time of pricing. Using a famous result by Merton and with some help from the path integral method, exact pricing formulas for European call and put options under the influence of virtual arbitrage returns (or intermediate deviations from economic equilibrium) are derived where only the final integration over initial arbitrage returns needs to be performed numerically. This result, which has not been given previously and is at variance with results stated by Ilinski *et al.*, is complemented by a discussion of the hedging strategy associated to a derivative, which replicates the final payoff but turns out to be not self-financing in the real world, but self-financing when summed over the derivative's remaining life time. Numerical examples are given which underline the fact that an additional positive risk premium (with respect to the Black-Scholes values) is found reflecting extra hedging costs due to intermediate deviations from economic equilibrium.

PACS. 02.50.Ey Stochastic processes – 05.40.Jc Brownian motion – 89.90.+n Other topics of general interest to physicists (restricted to new topics in section 89)

1 Introduction

The pricing and hedging of derivatives is a major task for financial institutions [23] and has become an increasingly popular topic in statistical physics [13,15]. Derivatives are sometimes also called contingent claims, as the buyer of the derivatives is entitled to receive a certain payoff up to (or at) some future time T, the time of expiry, dependent on the price S of a so-called "underlying" security (say a stock) within a certain time interval between today and T or at time T. The simplest case is a so-called European call (put) option which gives the buyer the right to buy (sell) an underlying security (*e.g.* a stock) at a certain time T in the future for a fixed price K (the strike price). These options are also called "plain vanilla options" for their simplicity (as common as the vanilla ice cream flavor).

The classical result of Black and Scholes [1] on option pricing which revolutionized the world of finance and still forms the foundation for most of modern research, is based on the existence of an equilibrium, generally called

"absence of arbitrage", *i.e.* the impossibility of a profit "for nothing".

The use of the no-arbitrage assumption for pricing purposes is nicely elucidated in a standard text book like [23] where simple pricing equations for forward contracts are derived from optimization arguments. If *e.g.* the forward price F at time t for buying or selling (assuming no bid/offer spreads and transactions costs) a non-dividend paying security S at a later time T were less than $S \exp(r(T - t))$, then a riskless profit could be obtained in the following way: at time t, one enters into a forward contract to buy the security for F at time T, and one short sells the security (*i.e.* one borrows the security from somebody else and sells it, assuming no fees for simplicity) and puts the proceeds on a deposit at the riskless interest rate r (assuming no credit risk); at time T, one receives the security from the forward contract thereby closing out the short position in the security (*i.e.* handing it over to the lender), while receiving the nominal amount plus interest from the deposit less the forward price paid: $S \exp(r(T - t)) - F > 0$. Likewise, an arbitrage is possible

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when $F > S \exp(r(T - t))$. As the information on either situation spreads in the market, the inequalities disappear, and the relation $F = S \exp(r(T - t))$ results. Now the no-arbitrage assumption anticipates the equality to hold right from the beginning, thus implying that arbitrage opportunities disappear infinitely fast. Now many trading activities are motivated exactly by the fact, that this is not the case, but that arbitrage returns exist in the market for a short time $\tau_{\text{arbitrage}} > 0$. After this time, the information on arbitrage opportunities has reached enough market participants to make them disappear.

The absence of arbitrage assumption paves the way for one of the fundamental pillars of mathematical finance which is the theorem by Harrison and Pliska [12]. In fact whenever markets are complete (*i.e.* when any derivative can be hedged by a self-financing strategy, which is a more restrictive statement than absence of arbitrage), then there is a unique equivalent martingale measure for the underlying security and *vice versa* (see [25] for an introductory discussion on martingale theory). A stochastic process X_t is a martingale with respect to the measure Q if and only if $E_{\text{Q}}[|X_t|] < \infty$

$$X_t = E_{\text{Q}}[X_s | \mathcal{F}_t], \qquad s \geq t \tag{1}$$

where $\mathcal{F}_t$ is the filtration at time t, *i.e.* the information accumulated until time t. This rather technical statement is the basis for risk-neutral valuation: Derivatives can be priced in a world where all yields are equal to the risk-free interest rate (minus dividend yields etc.).

Apart from the dynamic deviations from the no-arbitrage situation discussed above, there is a large literature on serious drawbacks of the Black-Scholes model itself which is classically used to implement no-arbitrage pricing of options, *i.e.* that price returns evolve according to Brownian motion with constant drift and volatility. Empirical studies of return distributions in fact show volatility clustering and fat tails [4,5,7,14], and so real price changes appear to be more efficiently modelled by truncated Lévy processes (TLP) [5,6]. However, rational option pricing using the martingale approach appears to be still working (see [8] and Refs. therein), so the main theme of the Black-Scholes method, *i.e.* the possibility to set up a self-financing hedging strategy (a notion to be explained below) seems to hold. Moreover, studies on autocorrelations of price changes or on the distributions of price changes themselves (taken for different time scales) demonstrate a crossover to Gaussian dynamics after a certain time scale which might vary from several minutes to days (depending on market liquidity) [7,14].

In the present work, this time scale is proposed to be proportional to $\tau_{\text{arbitrage}}$. In this sense, fat tails of distributions of price changes are a signature of intermediate arbitrage opportunities on short time scales (compared to $\tau_{\text{arbitrage}}$). The implications for pricing options that are not very short-lived seem to be that intermediate arbitrage opportunities may be modelled as deviations from Brownian motion, and thus as perturbative effects. One way to treat these deviations are stochastic volatility (SV) models (see for a review in the context of option pricing [9]).

The present work complements these models and gives an alternative approach which modifies the drift of the asset price process rather than its volatility.

If one tries to get rid of the drawbacks of the Black-Scholes model by dynamic parameters which are not directly tradable such as in SV models or in the approach discussed below, one encounters a new problem: as opposed to complete markets defined above, a self-financing strategy using traded instruments ceases to exist. In general, any hedging strategy can only reduce the risk inherent in the final payoff to an intrinsic component [2,13]. Technically, this leads to more than one equivalent martingale measure [12]. Ambiguity for derivatives pricing is handled by introducing additional constraints on the hedging strategy, *e.g.* minimizing the expected squared cost for the remaining life time of the option while exactly replicating the final payoff (local risk-minimization) or minimizing the expected squared net loss at the time of maturity of the option (mean-variance hedging) [2,3,16] (see also [13,15] for a physicist's approach). For the model presented in this work, a very specific method is proposed in order to select an equivalent martingale measure which satisfies both constraints simultaneously.

The issue of option pricing in incomplete markets has become a matter of practical interest recently, in particular with the increasing importance of credit derivatives. As opposed to conventional derivatives which cover market risks, there is not a large underlying market of actively traded credit risk instruments in every credit risk category. As opposed to a stock, *e.g.* a loan is usually not traded.

Considering fully complete and incomplete markets as extreme cases of real markets, one is naturally forced to ask for crossover effects or transitions between the two regimes. A possible answer to this question might be given in terms of a dynamic model which considers market incompleteness in terms of fluctuations around an economic equilibrium characterizing a complete market.

An important step in this direction has been given by Ilinski and Stepanenko [10] and Ilinski [11] who assume the existence of intermediate, "virtual" arbitrage returns x_t, which appear and disappear over a certain time scale which may be identified with $\tau_{\text{arbitrage}}$ mentioned above. In fact, Ilinski *et al.* intend to treat arbitrage effects as a perturbation to the usual Black-Scholes risk-free rate r, which gives the yield on all investments in a risk-neutral world. More specifically, the Black-Scholes risk-free rate is split into a constant part r^0 and an arbitrage return x_t according to $r_t = r^0 + x_t$. This model is different from stochastic volatility models: it modifies the (risk-adjusted) drift of the asset price process rather than its volatility. In principle there appears no reason to favor one approach over the other. In fact, preliminary results from simulations of the stochastic drift model *versus* the stochastic volatility model by Stein & Stein [17] seem to point in this direction. Detailed results on this comparison will be published elsewhere [21]. The stochastic drift approach discussed here has the advantage that it can be mapped to models with stochastic interest rates (see below).

Ilinski *et al.* [10,11] present two approaches to calculate option prices. In [10], a perturbative method is given based on the classical Black-Scholes equation where the constant risk-free rate r is replaced by $r^0 + x_t$ where r^0 is constant and x_t is random. This equation is then iterated to second order in x_t and averaged over with respect to x_t. The results obtained for European call and put options are not reproduced by our exact calculation. We come back to this difference at the end of Section 5 which explains why our results are reasonable for the specific dynamics of the arbitrage return used here. The observed difference just mentioned leads us to suggest that either the method or some further approximations made in [10] are not correct. In a second paper [11] Ilinski *et al.* derive a fully deterministic Black-Scholes type PDE that depends both on the current level of the security price S_t and the arbitrage return x_t. This equation is not further evaluated. In Section 3, we show that it contains a misprint. We add as a remark that the issue of measure change and the construction and selection of an equivalent martingale measure which is fundamental to option pricing (see *e.g.* [25]) is not addressed.

Therefore, in this article a different route is proposed. First, the arbitrage return x_t is considered as a part of a stochastic interest rate dynamics for the risk-free rate r_t in a virtual world (where arbitrage returns are directly observable). Essentially, one is led to a Black-Scholes type equation for a derivative depending on two state variables, the security price S_t and the arbitrage return x_t which is derived from the risk-free rate in the virtual world according to $r_t = r^0 + x_t$. The constant part r^0 is supposed to be the risk-free rate in the real world which consequently is assumed to be constant. The reason for the latter simplification is the later comparison with the Black-Scholes pricing formulas. The implementation of x_t as a part of a fictitious interest rate process leads to a stochastic drift for the asset process S_t (with respect to the particular martingale measure chosen, for details see below) and thus couples the dynamics of x_t to S_t. As the arbitrage return is an intermediate phenomenon on time scales shorter than the time to expiry, we follow Ilinski *et al.* by enforcing the boundary condition at the time of expiry of the option that the arbitrage return should disappear. This constraint may be relaxed, however, if one allows for the possibility that any hedging strategy might not replicate (*i.e.* provide for) the final payoff of the option. Nonetheless, we stay with this constraint (also in order to compare with the results of Ilinski *et al.* [10,11]). However, we do not implement this condition into the payoff function of the option like Ilinski [11], but into the average over arbitrage returns. This procedure allows us to use a famous result by Merton on options in a stochastic interest rate environment [26]. It is not meant to imply that intermediate arbitrage returns can be thought of as the random part of real interest rates. After averaging, we obtain a previously unknown exact result for European claims under the influence of virtual arbitrage. (The pricing of American claims which may be exercised prior to their maturity is possible in principle using a "tree" procedure [23], *i.e.* a

scheme based on discrete probabilities and discrete time). These exact pricing formulas differ significantly from the results obtained by Ilinski *et al.* [10].

The outline of this paper is as follows: In the next section, we present the route from the Black-Scholes model to a non-equilibrium market model, taking up the idea of intermediate ("virtual") arbitrage by Ilinski and Stepanenko. The third section will show how the effect of arbitrage returns on option pricing can be considered in terms of a stochastic interest rate environment in a virtual world. In Section 4, European call and put options are valued in the presence of virtual arbitrage returns. In Section 5, the issue of a replicating hedging strategy both in the virtual and real world and the selection of an equivalent martingale measure is addressed. Some explicit numerical pricing examples are given and their difference to the classical Black-Scholes results are explained in Section 6. In the final section, the results are briefly discussed.

2 From the Black-Scholes model to the dynamics of arbitrage returns

Let us briefly review the Black-Scholes analysis in order to motivate the notion of arbitrage returns, following [10]. The model for a one security market is given by

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t^1 \tag{2}$$

where S_t is the security price, μ the drift, and $\mathrm{d}W_t^1$ a Wiener process. It may be motivated from the fact that $\ln(S_{i+1}/S_i)$, where $i+1$ and i denote discrete points in time, performs a random walk [23]. Now the price of a derivative $V_t = V(S_t, t)$ whose payoff is contingent on the security price S_T at some future time T can be determined by setting up a portfolio Π_t consisting of the derivative V_t and a position $-\Delta$ of the security S_t:

$$\Pi_t = V_t - \Delta S_t. \tag{3}$$

Then if $\Delta = \frac{\partial V}{\partial S}$, where $S = S_t$, this portfolio is riskless as uncertainties arising from the Wiener process are eliminated which can be seen by evaluating $\mathrm{d}\Pi_t$ using Ito's lemma. Therefore, the portfolio is known to grow at the risk-free rate, *i.e.*

$$\mathrm{d}\Pi_t = r\Pi_t \mathrm{d}t \tag{4}$$

For constant interest rates, equating expressions for $\mathrm{d}\Pi_t$ gives the Black-Scholes partial differential equation (PDE):

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \tag{5}$$

Specifying a certain boundary condition to this equation representing the option payoff at the time of maturity completes the usual Black-Scholes pricing problem. As a reminder, let us note that the drift μ which was introduced in the market model equation (2) is absent from the pricing equation (5).

The idea of arbitrage returns may be motivated by assuming that in the presence of arbitrage opportunities, the true return of the portfolio Π_t is not equal to the constant risk-free interest rate r, but might be less or more than that. Following [11], an arbitrage return x_t is now introduced by substituting for r

$$r_t = r^0 + x_t \tag{6}$$

where x_t is assumed to follow the dynamics of a decay process with random noise:

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = -\lambda x_t + \eta_t \tag{7}$$

where η_t is characterized by:

$$\langle \eta_t \rangle = 0, \qquad \langle \eta_t \eta_{t'} \rangle = \Sigma^2 \delta(t - t'). \tag{8}$$

As to the nature of η_t further complications are discussed in [10], but they are not important for our analysis. Basically, a stochastic component x_t as been added to the constant risk-free rate r_0. The question now is: How does the process for the arbitrage return x_t affect the price of a derivative?

Substituting equation (6) for risk-free rate r in the standard Black-Scholes PDE, Ilinski and Stepanenko simply proceed and derive the following PDE:

$$\mathcal{L}_{\mathrm{BS}} V = x_t \left(V - S \frac{\partial V}{\partial S} \right) \tag{9}$$

where $\mathcal{L}_{\mathrm{BS}}$ is the operator from the standard Black-Scholes PDE, $\mathcal{L}_{\mathrm{BS}} V = 0$, for $r = r^0$. We will clarify below that the replacement $r \to r^0 + x_t$ in fact is equivalent to introducing an interest rate process $r_t = r^0 + x_t$ in a virtual world where tradable instruments dependent on this interest rate exist.

The specific origin of intermediate arbitrage returns and market incompleteness is assumed to be contained in the parameters λ which sets the time scale for deviations from market equilibrium and Σ which gives a measure for the arbitrage returns themselves. Specific values are discussed in Section 6. Of course, in reality transaction costs which are neglected here for simplicity might effectively destroy small arbitrage returns.

In the next section, a different approach is presented in the framework of standard option pricing theory which allows to study the influence of intermediate deviations from financial equilibrium (as defined by the no-arbitrage assumption) on derivative pricing.

3 Derivatives in the presence of arbitrage opportunities: A mapping to option pricing theory with stochastic interest rates

The way the arbitrage return x_t has been introduced in the last section, in particular that the portfolio Π_t grows at the rate $r^0 + x_t$, allows for a mapping to option pricing theory with stochastic interest rates. We will call $r_t = r^0 + x_t$

an interest rate in a virtual world, but do not insinuate that the arbitrage return is a part of the real interest rate. This virtual world will serve as a stage where known results can be used, but finally these results need to be projected to the real world. Let us for the moment assume, that this virtual world can be set up. The justification for its use will be delayed to Section 5.

Let us first review the PDE approach to option pricing with stochastic interest rates. The stochastic nature of a (short) interest rate r_t is usually taken into account by stating a stochastic differential equation (SDE) as follows:

$$\mathrm{d}r_t = \rho(r_t, t)\mathrm{d}t + \Sigma(r_t, t)\mathrm{d}W_t^2. \tag{10}$$

The parameters ρ and Σ specify drift and volatility respectively, and may depend on r_t and t. The drift specifies the deterministic ("trend") component of the interest rate dynamics whereas the volatility describes the stochastic fluctuations. The increment $\mathrm{d}W_t^2$ is a Wiener process. The general PDE for a derivative $V = V(S, r, t)$ dependent on $S = S_t$ and $r = r_t$ can be found in the literature [23]. Assuming for simplicity no correlations between the Wiener processes $\mathrm{d}W^1$ from equation (2) and $\mathrm{d}W^2$ and suppressing the functional dependence of Σ and ρ, the PDE is given by:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + S \frac{\partial V}{\partial S}(\mu - \lambda_1 \sigma) - rV$$
$$+ \frac{\Sigma^2}{2} \frac{\partial^2 V}{\partial r^2} + \frac{\partial V}{\partial r}(\rho - \lambda_2 \Sigma) = 0. \tag{11}$$

The parameters λ_i, $i = 1, 2$ are known as the market prices of risk for the security S and the risk-free rate r. They can be obtained by finding the change of measure which makes the respective discounted price process a martingale [25]. For a non-dividend paying security governed by equation (2), $\lambda_1 = (\mu - r)/\sigma$. Incidentally, if r is constant, one recovers the Black-Scholes PDE equation (5). As r is not a tradable security, a tradable interest rate instrument is needed, e.g. a zero bond with maturity T whose price at time t is $P(t, T)$ and which promises to pay one monetary unit at time T. In fact, one is left with a residual freedom of choosing λ_2 [24]. We will return to this issue instantly. Let us now restrict the drift ρ_r to a mean-reverting form:

$$\rho = \rho(r_t) = a - \lambda r_t. \tag{12}$$

Moreover, let us suppose that $\Sigma(r_t, t) = \Sigma = \text{const}$. Let us further assume that

$$r_t = r^0 + x_t. \tag{13}$$

Then after transforming from r to x the PDE for the derivative price V reads as:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + S \frac{\partial V}{\partial S}(r^0 + x) - (r^0 + x)V$$
$$+ \frac{\Sigma^2}{2} \frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial x}(a - \lambda r^0 - \lambda_2 \Sigma - \lambda x) = 0 \tag{14}$$

As pointed out in [24], it is not possible to separate the market price of risk λ_2 from the difference $\tilde{a} = a - \lambda_2 \Sigma$. Let us now return to the process for the arbitrage return x_t assumed in the last section, equation (7). Under the martingale measure for the discounted zero bond price, the process for x_t obtained from equation (10) together with (12) and (13) using standard techniques [25] is given by the SDE:

$$\mathrm{d}x_t = \left(\tilde{a} - \lambda r^0 - \lambda x_t \right) \mathrm{d}t + \Sigma \mathrm{d}\tilde{W}_t^2. \qquad (15)$$

The change to a measure $\tilde{W}_t^2$ that makes the discounted zero bond price a martingale of course amounts to a choice as there is no such instrument in the real world. This means that there is no unique martingale measure in terms of real world instruments so we are necessarily forced to choose one. One possibility is to require that the process equation (15) under the martingale measure is mean-reverting to zero: deviations from economic equilibrium should disappear to zero. This means that $\tilde{a} - \lambda r^0 = 0$, making equation (15) identical to the corresponding expression in equation (7).

Then equation (14) has a similar but not the same form as equation (3) in [11] (the last term should read $-\lambda x \frac{\partial V}{\partial x}$ instead of $+\lambda \frac{\partial x V}{\partial x}$),

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + S \frac{\partial V}{\partial S}(r^0 + x)$$
$$- (r^0 + x)V + \frac{\Sigma^2}{2} \frac{\partial^2 V}{\partial x^2} - \lambda x \frac{\partial V}{\partial x} = 0 \quad (16)$$

and the process for the arbitrage return becomes

$$\mathrm{d}x_t = -\lambda x_t \mathrm{d}t + \Sigma \mathrm{d}\tilde{W}_t^2 \qquad (17)$$

yielding the consistency of the no-arbitrage approach in the virtual world with the arbitrage dynamics proposed in equation (7). The security price dynamics becomes

$$\mathrm{d}S_t = (r^0 + x_t)S_t \mathrm{d}t + \sigma S_t \mathrm{d}\tilde{W}_t^1 \qquad (18)$$

with respect to the martingale measure. This equation basically couples arbitrage returns to security price dynamics under the chosen martingale measure. Thus incompleteness is introduced here in terms of stochastic drift as mentioned above.

Of course, the dynamics of real interest rates are not the same as the dynamics proposed here for a virtual world. The relaxational time scale $1/\lambda$ originating from the disappearance of virtual arbitrage returns is much shorter than a time scale of mean reversion for real interest rates. As will become clear in the next section, our zero bond price in the virtual world will approach a real (constant interest rate) bond price in the limit of infinitely fast relaxation dynamics for the arbitrage return x_t. One might object at this point, that we have assumed a hedging strategy in the virtual world which does not exist in the real world (x_t cannot be hedged). Indeed, the hedging strategy in the virtual world expressed in terms of the security S_t and a real world cash bond $B_t^0 = \exp(-r^0 t)$

leaves us with an extra amount arising from the dynamics of x_t, and is therefore not self-financing in terms of real world instruments. We will address this issue in more detail in Section 5.

In order to complete the pricing problem in the virtual world, equation (16) requires the boundary condition, *e.g.* for a European claim (which is exercised or not exactly at time T). It must be chosen as [11]

$$V(t, S, r)|_{t=T} = X\delta(x) \qquad (19)$$

where X is the final payoff (in the real world) depending on $S(T)$ and $r = r^0 + x$. Then the option price in the real world may be calculated as an average over the initial arbitrage return as follows:

$$\bar{V}(t, S, r^0) = \int_{-\infty}^{\infty} \mathrm{d}x V(t, S, r)\tilde{p}(x) \qquad (20)$$

where $\tilde{p}(x)$ is a probability density function chosen according to the dynamics of x_t to be discussed below.

However, we will not proceed to solve the PDE, but remember that according to the Feynman-Kac lemma [18,19]

$$V(t, S, r) = E_{\mathrm{Q}} \left[\mathrm{e}^{-\int_t^T \mathrm{d}s r_s} X\delta(x_T) | x_t = x, r^0, S_t \right]. \quad (21)$$

Now it is easy to show (*e.g.* by using the path integral approach [20]) that

$$E_{\mathrm{Q}} \left[\mathrm{e}^{-\int_t^T \mathrm{d}s r_s} X\delta(x_T) | x_t = x, r^0, S_t \right]$$
$$= E_{\mathrm{Q}} \left[\mathrm{e}^{-\int_t^T \mathrm{d}s r_s} X | x_t = x, r^0, S_t; x_T = 0 \right]$$
$$\times p(x_T = 0 | x_t = x, r^0, S_t) \qquad (22)$$

where $p(x_T = 0 | x_t = x, r^0, S_t)$ is the conditional probability density function for $x_T = 0$ given $x_t = x, r^0, S_t$. The last equation allows us to utilize results from the literature on option pricing theory with stochastic interest rates. In fact, we will solve the pricing problem for the actual payoff function X in an interest rate environment where the short rate process in the virtual world r_t starts from $r^0 + x$ at time t and comes back to r^0 at the time of maturity T, or put otherwise where $x_t = x$ and $x_T = 0$.

The average as given in equation (20) will then be performed in a different way. From the constraint $\bar{V}(t, S, r_0)|_{t=T} = X$, it clear that

$$\tilde{p}(x) = \frac{p(x)}{p(0)} \qquad (23)$$

where $p(x)$ is the probability density function for the initial value x of the arbitrage return. Using the fact that in our case $p(x_T = 0 | x_t = x, r_0, S_t) = p(x_T = 0 | x_t = x)$ and

$$p(x_t = x | x_T = 0) = p(x_T = 0 | x_t = x)\frac{p(x)}{p(0)} \qquad (24)$$

one may rewrite the average in equation (20) as follows:

$$\bar{V}(t,S,r^0) = \int_{-\infty}^{\infty} dx V(t,S,r;x_T=0)p(x_t=x|x_T=0) \tag{25}$$

where

$$V(t,S,r;x_T=0) = E_Q\left[e^{-\int_t^T ds r_s}X|x_t=x,r^0,S_t;x_T=0\right] \tag{26}$$

and where $p(x_t=x|x_T=0)$ is the conditional probability density for the arbitrage return at time t to be equal to x given that its value at $T>t$, the time of expiry, is zero. Its explicit form will be discussed in the next section. In fact, as $t=T$, one obtains $p(x_t=x|x_T=0) = \delta(x)$ as required.

4 Valuation of European call and put options in the presence of virtual arbitrage opportunities

Considering the arbitrage return as a part of the stochastic interest rate r_t in our virtual world, we can now draw upon a classical result of Merton [26] in order to derive formulas for European call and put options. In fact, instead of solving equation (16) together with equation (19) for $X = \max(S-K;0)$ (or $X = \max(K-S;0)$) for call or put options respectively, we consider the security price dynamics equation (2) together with the following SDE for the price of a zero bond:

$$d_t P(t,T) = P(t,T)\left(\mu_P(t,T)dt + \sigma_P(t,T)dW_t^2\right). \tag{27}$$

Now assuming that $\sigma_P(t,T)$ is a known function of t and T, the price of a European call (and put) option c (and p) at time t which expires at time T with strike price K is given by [26]:

$$c = SN(d_1) - P(t,T)KN(d_2) \tag{28}$$
$$p = P(t,T)KN(-d_2) - SN(-d_1) \tag{29}$$

where $N(x)$ is the cumulative normal distribution and

$$d_1 = \frac{\ln(S/K) - \ln(P(t,T)) + \hat{\sigma}^2(T-t)/2}{\hat{\sigma}\sqrt{T-t}}$$
$$d_2 = d_1 - \hat{\sigma}\sqrt{T-t}$$
$$\hat{\sigma}^2(T-t) = \int_t^T ds\left(\sigma^2 + \sigma_P^2(s,T) - 2\rho_{SP}\sigma\sigma_P(s,T)\right) \tag{30}$$

The parameter σ is the volatility of the security, and ρ_{SP} is the instantaneous correlation between the stock and zero bond prices which for simplicity we assume to be zero as above. Let us now connect to the stochastic interest rate dynamics r_t. If the process for $P(t,T)$ is derived from the process for r_t using Ito's lemma, one obtains the following

dependence of the bond volatility on the parameters of the process for r_t:

$$\sigma_P(t,T) = \Sigma \frac{1}{P(t,T)}\frac{\partial P}{\partial r} \tag{31}$$

where Σ is the short rate volatility. For the specific short rate dynamics chosen in equations (10) and (12), the zero bond price can be calculated explicitly as a function of t, T, the current short rate level r and the model parameters. Concerning the latter ones, the drift of the short rate process has to be risk-adjusted by the market price of risk giving $\tilde{a}$ as mentioned above. Let us now calculate the bond price $P(t,T)$. Using the fact that

$$P(t,T) = E_Q\left[e^{-\int_t^T ds r_s}|r_0\right]$$
$$= e^{-r^0(T-t)}E_Q\left[e^{-\int_t^T ds x_s}|x_t=x\right] \tag{32}$$

one obtains from the dynamics of x_t equation (17) (after a tedious calculation given in the Appendix using the path integral approach [20]) the following result:

$$P(t,T) = \exp\left(-\left(r^0 - \frac{\Sigma^2}{2\lambda^2}\right)(T-t)\right.$$
$$\left.-\frac{1}{\lambda}\tanh\left(\frac{\lambda(T-t)}{2}\right)\left(x + \frac{\Sigma^2}{\lambda^2}\right)\right). \tag{33}$$

Performing the differentiation in equation (31), one obtains:

$$\sigma_P(t,T) = -\frac{\Sigma}{\lambda}\tanh\left(\frac{\lambda(T-t)}{2}\right). \tag{34}$$

Now, the restriction $\tilde{a} = \lambda r^0$ which makes the drift of the process for the arbitrage return x_t be equal to $-\lambda x_t$ (under the martingale measure), gives the desired asymptotics of the zero bond price. In fact, as $\lambda \to \infty$, which can be interpreted as an infinitely fast disappearance of virtual arbitrage returns, it reads as

$$\lim_{\lambda\to\infty} P(t,T) = e^{-r^0(T-t)}, \tag{35}$$

the zero bond price for a constant risk-free rate r^0. Therefore the restriction on $\tilde{a}$ mentioned above, and thus our choice of the martingale measure is reasonable also from the viewpoint of correct zero bond price asymptotics. Let us know turn to the evaluation of the modified security price volatility $\hat{\sigma}$. Evaluating the integral in equation (30), one obtains:

$$\hat{\sigma}^2 = \sigma^2 + \frac{\Sigma^2}{\lambda^2}\left(1 - \frac{2}{\lambda(T-t)}\tanh\left(\frac{\lambda(T-t)}{2}\right)\right). \tag{36}$$

Likewise, in the limit $\lambda \to \infty$, the contribution to virtual arbitrage returns disappears and one recovers the "bare" security price volatility:

$$\lim_{\lambda\to\infty} \hat{\sigma} = \sigma. \tag{37}$$

The asymptotic equations equations (35) and (37) assure that in the case of infinitely fast vanishing arbitrage returns the Black-Scholes formulas (for a constant risk-free rate r^0) are recovered from equation (28). When the option approaches maturity, there is the following expansion of $\hat{\sigma}^2$:

$$\hat{\sigma}^2 = \sigma^2 + \frac{\Sigma^2}{12}(T-t)^2 + \mathcal{O}\left((T-t)^3\right). \qquad (38)$$

Now as the option price in our virtual world is fixed in terms of the parameters of the arbitrage return process, we need to turn to the explicit evaluation of the average carried out in equation (25). For an Ornstein-Uhlenbeck process x_t given by equation (17) it is well known [27], that the transition probability to go from x' at time 0 to x at time t is given by:

$$p(x_t = x|x_0 = x') = \sqrt{\frac{\lambda}{\pi \Sigma^2}} \left(1 - e^{-2\lambda t}\right)^{-1/2}$$
$$\times \exp\left(-\frac{\lambda}{\Sigma^2} \frac{(x - x'e^{-\lambda t})^2}{(1 - e^{-2\lambda t})}\right) \quad (39)$$

What is needed however in our case, is $p(x_t = x|x_T = 0)$ for $T \geq t$ which is obtained from equation (39) as follows:

$$p(x_t = x|x_T = 0) = p(x_T = 0|x_t = x)\frac{p(x)}{p(0)} \qquad (40)$$

where $p(x)$ is the probability density for x which is obtained as a limit probability density form equation (39) as $t \to \infty$:

$$p(x) = \sqrt{\frac{\lambda}{\pi \Sigma^2}} \exp\left(-\frac{\lambda}{\Sigma^2}x^2\right). \qquad (41)$$

The final expression for the transition probability thus reads:

$$p(x_t = x|x_T = 0) = \sqrt{\frac{\lambda}{\pi \Sigma^2}} \left(1 - e^{-2\lambda(T-t)}\right)^{-1/2}$$
$$\times \exp\left(-\frac{\lambda}{\Sigma^2}x^2 \frac{1}{\left(1 - e^{-2\lambda(T-t)}\right)}\right). \qquad (42)$$

It has all the desired features needed. Using the following representation of Dirac's delta function:

$$\lim_{n \to \infty} n e^{-\pi n^2 x^2} = \delta(x) \qquad (43)$$

one obtains both for the limit of infinitely rapid disappearance of arbitrage returns

$$\lim_{\lambda/\Sigma^2 \to \infty} p(x_t = x|x_T = 0) = \delta(x) \qquad (44)$$

and for the limit $t \to T$ of approaching the option's time of maturity

$$\lim_{t \to T} p(x_t = x|x_T = 0) = \delta(x). \qquad (45)$$

In both cases, one expects arbitrage returns to disappear. Next, the average over virtual arbitrage returns in equation (25) is carried out explicitly for a European call option (for $\bar{V} = \bar{c}$) as

$$\bar{c}(t, S, r_0) = S \int_{-\infty}^{\infty} dx N(d_1) p(x(t) = x|x(T) = 0)$$
$$- K \int_{-\infty}^{\infty} dx P(t, T) N(d_2) p(x(t) = x|x(T) = 0) \qquad (46)$$

and a European put option (for $\bar{V} = \bar{p}$) as

$$\bar{p}(t, S, r_0) = K \int_{-\infty}^{\infty} dx P(t, T) N(-d_2) p(x(t) = x|x(T) = 0)$$
$$- S \int_{-\infty}^{\infty} dx N(-d_1) p(x(t) = x|x(T) = 0) \qquad (47)$$

where $P(t, T)$ is given in equation (33). The integrations with respect to x cannot be performed analytically. However, the integrands decrease sufficiently fast to zero as $x \to \pm\infty$, so that a numerical integration can be be easily performed.

It is obvious from intuition that the pricing formulas equations (46) and (47) contain the fundamental time scale $\tau_{\mathrm{arbitrage}} = 1/\lambda$. In fact, one can introduce the following scaled variables:

$$u = \lambda(T - t) = \frac{(T - t)}{\tau_{\mathrm{arbitrage}}}$$
$$r_\lambda = \frac{r}{\lambda}$$
$$x_\lambda = \frac{x}{\lambda}$$
$$\hat{\sigma}_\lambda = \frac{\hat{\sigma}}{\sqrt{\lambda}}$$
$$\sigma_\lambda = \frac{\sigma}{\sqrt{\lambda}}$$
$$\Sigma_\lambda = \frac{\Sigma}{\lambda^{3/2}}. \qquad (48)$$

Then λ can be eliminated from the pricing formulas. The parameters in equation (30) can expressed in terms of the scaled variables of equation (48):

$$d_1 = \frac{\ln(S/K) - \ln(P(u)) + \hat{\sigma}_\lambda^2 u/2}{\hat{\sigma}_\lambda \sqrt{u}}$$
$$d_2 = d_1 - \hat{\sigma}_\lambda \sqrt{u}$$
$$\hat{\sigma}_\lambda^2 = \sigma_\lambda^2 + \Sigma_\lambda^2 \left(1 - \frac{2}{u}\tanh\left(\frac{u}{2}\right)\right) \qquad (49)$$

where $P(u)$ is given by:

$$P(u) = P(t, T) = \exp\left(-\left(r_\lambda^0 - \frac{1}{2}\Sigma_\lambda^2\right)u\right.$$
$$\left. - \tanh\left(\frac{u}{2}\right)\left(x_\lambda + \Sigma_\lambda^2\right)\right). \qquad (50)$$

5 Replicating hedging strategies

The issue of hedging strategies in the virtual and the real world mentioned above will now be addressed. The fact that there is no instrument in the real world to hedge intermediate arbitrage returns leads us to conjecture that a hedging strategy might not be self-financing.

To be specific, let us denote a cash bond in our virtual world as follows:

$$B_t = \exp\left(\int_0^t \mathrm{d}s r_s\right). \tag{51}$$

It monitors the temporal evolution of the value of an initial cash deposit $B_0 = 1$ which earns the instantaneous interest rate r_s. Let us further introduce the cash bond in the real world

$$B_t^0 = \exp(r^0 t) \tag{52}$$

and as a further abbreviation (which may be termed the "arbitrage bond")

$$B_t^x = \exp\left(\int_0^t \mathrm{d}s x_s\right). \tag{53}$$

Evidently, one obtains:

$$B_t = B_t^0 B_t^x. \tag{54}$$

Taking B_t for the moment as a real cash bond, a self-financing strategy V_t consists of holding φ_t in the security S_t and ψ_t in the cash bond B_t such that

$$V_t = \varphi_t S_t + \psi_t B_t \Rightarrow \mathrm{d}V_t = \varphi_t \mathrm{d}S_t + \psi_t \mathrm{d}B_t \tag{55}$$

i.e. the value change $\mathrm{d}V_t$ is only due to price changes $\mathrm{d}S_t$ and $\mathrm{d}B_t$. For our security price model with stochastic interest rates in the virtual world, one can show that equation (55) holds [25]. Moreover $V_T = X$, *i.e.* the value of portfolio V equals the final payoff, *i.e.* it is replicating. So in terms of our fictitious cash bond B_t there is a self-financing, replicating strategy. In the real world, our strategy will remain replicating by construction (see Eqs. (20) to (25)). However, it will not be self-financing in terms of the real cash bond B_t^0 and the security price S_t, as can be seen by substituting for B_t in equation (55):

$$\begin{aligned}
dV_t &= \varphi_t \mathrm{d}S_t + \psi_t \mathrm{d}(B_t^0 B_t^x) \\
&= \varphi_t \mathrm{d}S_t + \psi_t B_t^x dB_t^0 + \psi_t B_t^0 \mathrm{d}B_t^x \\
&= \varphi_t \mathrm{d}S_t + \psi_t B_t^x dB_t^0 + \psi_t B_t^0 B_t^x x_t \mathrm{d}t \\
&= \varphi_t \mathrm{d}S_t + \psi_t B_t^x dB_t^0 + (V_t - \varphi_t S_t) x_t \mathrm{d}t. \tag{56}
\end{aligned}$$

The last step was to replace $\psi_t B_t^0 B_t^x = \psi_t B_t$ by $V_t - \varphi_t S_t$ using equation (55). The third term on the r.h.s of the last line accounts for extra costs or gains due to arbitrage opportunities. It is exactly equal to the instantaneous (positive or negative) arbitrage return earned on the delta hedge $V_t - \varphi_t S_t$. In fact, one has $\Delta = \varphi_t$, and therefore

$$\Pi_t = V_t - \varphi_t S_t \tag{57}$$

where Π_t is the delta hedge portfolio discussed in Section 2. The replacement $r \rightarrow r^0 + x_t$ introduced by Ilinski [10] gives rise to the same additional term in the hedging strategy V_t, if one considers the change $d\Pi_t$ as follows:

$$\mathrm{d}\Pi_t = (r^0 + x_t)\Pi_t \mathrm{d}t = r^0 \Pi_t \mathrm{d}t + x_t \Pi_t \mathrm{d}t. \tag{58}$$

The second term on the r.h.s of this equation is the source of additional intermediate profit and loss (p&l) during the hedging process. Therefore, we conclude that the replacement of Ilinski is completely equivalent to the introduction of a fictitious cash bond B_t or likewise an interest rate r_t as defined above, which ensures a self-financing hedging strategy in the virtual world.

The additional hedging costs or gains which arise in the real world are covered by an additional premium contained in the option price as obtained in equation (25) (with respect to the Black-Scholes price). This premium is positive in most cases as will be clarified below when numerical examples are discussed.

Finally, let us further back up the interpretation of $(V_t - \varphi_t S_t) x_t \mathrm{d}t$ as representing the differential p&l on the hedging strategy within the time interval $\mathrm{d}t$, using the following argument (whose formulation is borrowed from [14]). At time t, an option is sold at O_t in the real world, and using the premium the following portfolio $\langle V_t \rangle$ is bought:

$$\langle V_t \rangle = \langle \varphi_t \rangle S_t + \langle \psi_t B_t^x \rangle B_t^0 = O_t \tag{59}$$

where $\langle \ldots \rangle$ corresponds to an average over all paths $\{x_s\}_{s \in [t,T]}$. Furthermore, the change in wealth of the option seller within the time interval $[t, T]$ in the real world is given by:

$$\begin{aligned}
\Delta W = O_t &+ \int_t^T \langle \varphi_s \rangle \mathrm{d}S_s + \int_t^T \langle V_s - \varphi_s S_s \rangle r^0 \mathrm{d}s \\
&+ \int_t^T \langle V_s - \varphi_s S_s \rangle x_s \mathrm{d}s - X. \tag{60}
\end{aligned}$$

The first term is the option premium earned, the second term gives the cumulative gain by the trading the asset, the third one corresponds to the cost/gain of the cash bond position (used to finance the position in the asset or set side as excess cash respectively) which is proportional to the riskless rate r^0, and the fourth term is supposed to take into account the p&l due to virtual arbitrage. In fact, the fourth term can be added to third term giving an effective cost/gain of the cash bond position due to the effective rate $r^0 + x_t$. Finally the last term is the potential cash outflow due to the option's payoff. Now using $\mathrm{d}\langle V_t \rangle = \langle \varphi_t \rangle \mathrm{d}S_t + \langle (V_t - \varphi_t S_t)(r^0 + x_t) \rangle \mathrm{d}t$ one shows that:

$$\Delta W = O_t + \int_t^T \mathrm{d}\langle V_s \rangle - X = O_t - \langle V_T \rangle - \langle V_t \rangle - X = 0 \tag{61}$$

as $V_T = X$ by construction. As ΔW vanishes identically, $\overline{\Delta W^2}(t, S, r^0)$ (where the average is taken as in equation (25)) vanishes as well which implies that no intrinsic

risk remains over the remaining time to maturity of the option, and therefore no risk-minimization is necessary. The influence of virtual arbitrage is completely taken care of by the option premium. We see also that our hedging strategy in the real world is not self-financing at every time step but is self-financing when the time integral over remaining life time of the option is taken.

As discussed *e.g.* in [2], incomplete markets imply that there is no unique equivalent martingale measure any more. However, martingale theory may still be used if a supplementary constraint is added (see the discussion presented in the introduction) which then selects a particular martingale measure. In our case this choice has been implicitly made when the arbitrage return becomes part of a fictitious interest rate in the virtual world. In fact, both local risk (expected conditional squared cost) and replication risk (expected squared deviation of the terminal hedging portfolio to payoff) [3] are trivially minimized, *i.e.* zero. A detailed comparison of our approach to incomplete markets to the Föllmer-Schweizer approach [2] certainly deserves further study.

6 Some numerical results

In the following, some results are presented for two market situations, a rather incomplete market (Figs. 1 and 2) and a fairly complete market (Figs. 3 and 4). In the first case, the averaged prices $\bar{c}$ (and $\bar{p}$) , the Black-Scholes prices and the payoff functions at maturity are given, for parameters $\lambda = 10$, $T - t = 0.8$, $\Sigma = 2$, $\sigma = 0.2$, $K = 100$, $r^0 = 0.08$. The unit of time is 1 year, so $\lambda = 10$ corresponds to the rather long relaxation time $\tau_{\text{arbitrage}}$ of about 25 trading days, supposing a year of 250 trading days. $\Sigma = 2$ is inferred from a daily maximum variation of x_t of about 20% in absolute value (at 95% confidence level) according to the discretized Langevin equation:

$$\Delta x = x_{t+1} - x_t = -\lambda x_t \Delta t + X \Sigma \sqrt{\Delta t}. \qquad (62)$$

The random variable X is standard normally distributed. Taking $\Delta t = 1/250$, $X = 1.65$ representing the two-sided 95% confidence interval, $x_t = 0$ (as an initial value), one concludes

$$\Sigma = 9.58 \Delta x \qquad (63)$$

For various times to maturity $T - t$, Figures 3 and 4 presents differences of $\bar{c}$ (and $\bar{p}$) and the Black-Scholes prices for the choice of parameters $\lambda = 100$, $\Sigma = 0.4$, $\sigma = 0.2$, $K = 100$, $r^0 = 0.08$. $\lambda = 100$ corresponds to a relaxation time of 2 to 3 trading days, whereas $\Sigma = 0.4$ is inferred from a daily variation of x_t of about 4% in absolute value.

Let us now comment on the results. Focusing first on the qualitative behavior, over a reasonable range of the moneyness parameter $m = S/K$, the price of a European call or put option ($\bar{c}$ or $\bar{p}$ respectively) under the influence of virtual arbitrage is higher than the Black-Scholes value (see Figs. 1 and 2). The difference is more pronounced at
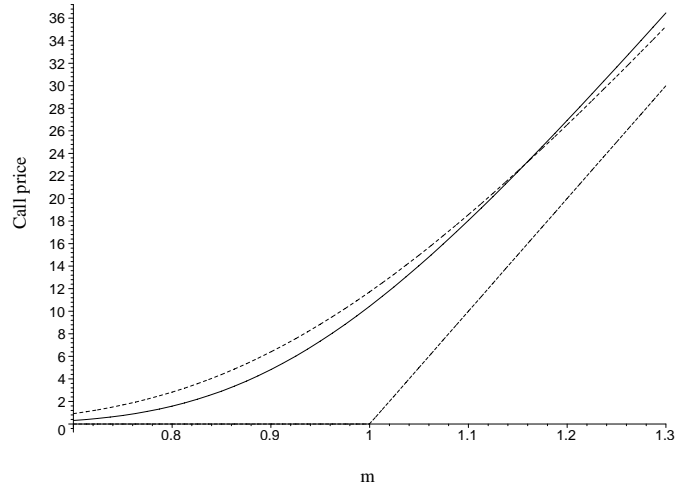

Fig. 1. The call option price as a function of moneyness $m = S/K$ (dashed curved line: with virtual arbitrage; solid line: Black-Scholes formula). The dashed straight line is the payoff function at maturity. Parameters: $\lambda = 10$, $T - t = 0.8$, $\Sigma = 2$, $\sigma = 0.2$, $K = 100$, $r^0 = 0.08$.
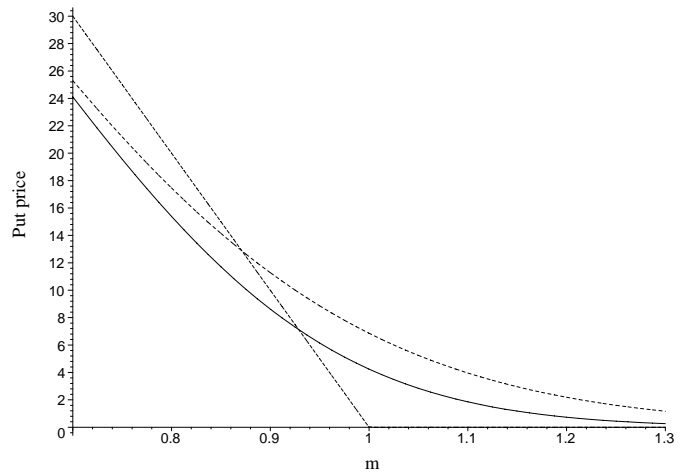

Fig. 2. The put option price as a function of moneyness $m = S/K$ (dashed curved line: with virtual arbitrage; solid line: Black-Scholes formula). The dashed straight line is the payoff function at maturity. Parameters: see Figure 1.

the point of maximum curvature which is around $m \simeq 1$ or below , whereas it decreases whenever $m < 1$ or $m > 1$ (see Figs. 3 and 4). For $m \gg 1$, the call option price is less than the Black-Scholes value. As the time to expiry increases the positive difference (except for $m \gg 1$) increases, and the maximum difference is shifted to lower values of m.

Leaving aside for the moment the negative difference appearing for call option at $m \gg 1$, it appears reasonable that the existence of virtual arbitrage returns causes the option price to be above the Black-Scholes value, as deviations from equilibrium in general lead to an increase in hedging costs, *i.e.* the costs for readjusting a replication portfolio which is supposed to provide for the final payoff of the option. This effect needs to be accounted for in the option premium. The fact that the absolute difference to
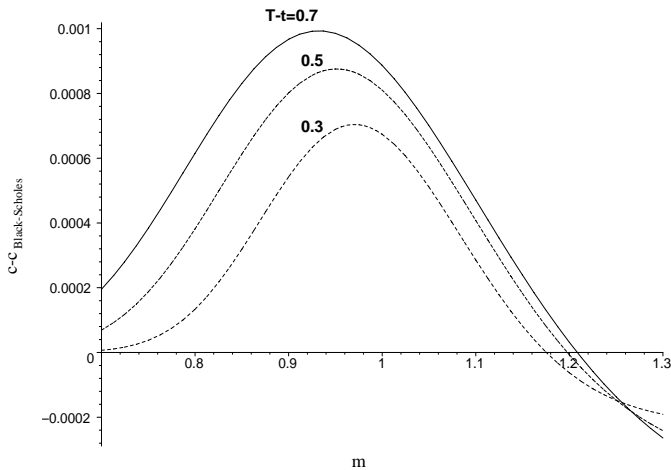
Fig. 3. Difference of the call price to Black-Scholes value (in absolute value) for various values $T - t$. Other parameters: $\lambda = 100$, $\Sigma = 0.4$, $\sigma = 0.2$, $K = 100$, $r^0 = 0.08$.
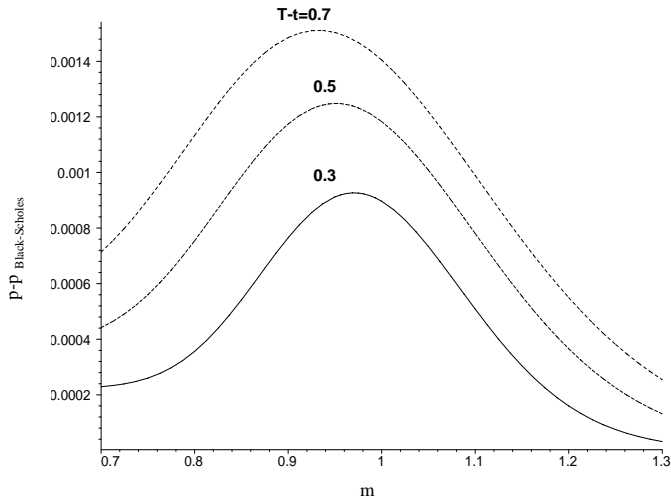

Fig. 4. Difference of the put price to Black-Scholes value (in absolute value) for various values $T - t$. Other parameters: see Figure 3.

the Black-Scholes result is the largest at the point of maximum curvature of the pricing function is understandable from the Γ ("gamma") risk point of view. Γ denotes the second derivative of the option price with respect the asset price S and gives a measure for the non-linear dependence of the option on the underlying asset. This non-linear risk inherent to options can be only be hedged by buying or selling other options. Any deviations from financial equilibrium due to arbitrage opportunities will affect both the option at hand and the options chosen for hedging. Moreover, the additional term arising in equation (56) leading to intermediate p&l during the hedging process is proportional to the delta hedge $V_t - \varphi_t S_t$ which is most relevant at the point of maximum Γ where the delta hedge is insufficient. Therefore, these numerical results are completely consistent with our mathematical discussion of the hedging strategy.

The influence of intermediate arbitrage returns grows as the time to expiry of the option increases on the scale of $\tau_{\text{arbitrage}}$ (see Figs. 3 and 4) (all other parameters being constant). Several deviations from equilibrium during the life time of an option seem to accumulate leading to a higher additional risk premium on the option price.

Returning to the issue of the negative difference for call options that are far in the money $m \gg 1$ in Figures 1 and 3, a possible explanation is an "overheated" market, where deviations from equilibrium tend to relax from the current asset price to a lower equilibrium price. This information is accounted for by pricing the option at a discount with respect to the Black-Scholes value at the current asset price: the market is expected to decrease to a lower price level.

Considering the quantitative differences between $\bar{c}$ (and $\bar{p}$) and the Black-Scholes prices for calls and puts, they are obviously more pronounced in an incomplete market (Figs. 1 and 2), than in a rather complete market where arbitrage returns are small and relax fast (3 and 4). The numerical analysis given here may be refined in various ways (according to the parameter dependences of the options prices) which is the subject of future work.

As opposed to our results, e.g. the first order correction to the Black-Scholes prices for calls given in [10] increases monotonously with moneyness m. Obviously, as our result is based on the same model as in [10] (see Sect. 5) and is exact (apart from the remaining integration over the initial arbitrage), some error is made in the perturbative treatment. Let us point out here again that the Ornstein-Uhlenbeck dynamics of arbitrage returns used here and the fact that the x_t gives the extra return on the delta hedge $V_t - \varphi_t S_t$, it is quite reasonable that the difference to the Black-Scholes price should show a maximum at the price level where the delta hedge fails.

7 Conclusion

Using the (Ornstein-Uhlenbeck type) relaxational dynamics for "virtual" arbitrage returns introduced in [10], we have derived closed formulas for simple ("plain vanilla") European calls and puts in the presence of arbitrage opportunities appearing and disappearing on an intermediate time scale $\tau_{\text{arbitrage}} = 1/\lambda$. This result which has not been derived previously is obtained using martingale option pricing theory for incomplete markets (in the sense of [2]), by making the arbitrage return process part of an interest rate process in a virtual world. The influence on option prices in the real world (in the presence of rapidly appearing and disappearing arbitrage opportunities) is taken into account by summing over the initial arbitrage return, and imposing the constraint that arbitrage is absent at the time of maturity of the option.

Comparing our work to [10,11], first, we consider the analysis given above as conceptually more clear as to where arbitrage-free pricing fails and where it does not. Therefore, in the present work a different route has been

proposed by introducing a second source of randomness in the derivative pricing problem (apart from the security S) right from the beginning. As a consequence, a two variable version of Ito's lemma must be used, giving a PDE equation for the derivative price in a virtual world which is finally summed over x_t to yield the real world price. Second, instead of making the constraint that arbitrage return should vanish at maturity a part of the payoff function in the virtual world as in [10], we enforce it when the average over virtual arbitrage return is taken. This procedure allows us to profit from Merton's classical result on option pricing in a stochastic interest rate environment [26] and to arrive at closed-form (up to a numerical integration over the initial arbitrage return which is easy to perform) pricing formulas for simple European call and put options.

Furthermore it has been shown that any hedging strategy will not be self-financing in the real world where the arbitrage return is not directly observable. However, on the average any intermediate costs arising during the hedging process are covered by an additional premium contained in the option price. In this sense, a hedging strategy can be found that is self-financing in a time average sense, *i.e.* when summed over the remaing life-time of the option. The derivation of pricing formulas rests crucially on the selection of a specific measure from a set of equivalent martingale measures that contains more than one element, due to intermediate market incompleteness which arises because of virtual arbitrage opportunities.

The present work may be extended in various directions. The relaxational dynamics of the arbitrage return may be considered to be more complicated as proposed here where it follows a simple Ornstein-Uhlenbeck process. However, additional model parameters introduce more sources of model error from the practitioner's point of view as each parameter has to calibrated to the market. Furthermore, the constraint $x_T = 0$, *i.e.* that arbitage returns should disappear at the time of maturity of the option, may be relaxed to allow for a hedging mismatch at maturity. This amounts to give up the constraint that the hedging strategy is replicating. The extension of this work to the case of correlations between the asset price S_t and the arbitrage return x_t is under way. Certainly, the comparison of the present model to stochastic volatility models deserves further study.

Appendix

Following [20] we propose to evaluate the expectation value

$$I = E_{\mathrm{Q}}\left[\mathrm{e}^{-\int_t^T \mathrm{d}s\, x_s}|x_t = x\right]. \tag{A.1}$$

The expectation value can be stated in terms of a quotient of path integrals as follows:

$$I =$$

$$\frac{\int_{x(t)=x}^{x(T)=0} \mathcal{D}x(s) \exp\left(-\frac{1}{2\Sigma^2}\int_t^T \mathrm{d}s \left(\frac{\mathrm{d}x(s)}{\mathrm{d}s}+\lambda x(s)\right)^2 - \int_t^T \mathrm{d}s\, x(s)\right)}{\int_{x(t)=x}^{x(T)=0} \mathcal{D}x(s) \exp\left(-\frac{1}{2\Sigma^2}\int_t^T \mathrm{d}s \left(\frac{\mathrm{d}x(s)}{\mathrm{d}s}+\lambda x(s)\right)^2\right)}$$

$$= \frac{X}{Y}. \tag{A.2}$$

Now the numerator and the denominator can be mapped to the propagator of the harmonic oscillator in the presence of an external field, and can thus be evaluated [28]. The expression for the numerator reads as

$$\begin{aligned}
X = &\sqrt{\frac{\lambda}{2\pi\Sigma^2 \sinh(a(T-t))}} \\
&\times \exp\left(\frac{\Sigma^2}{2\lambda^3}\left(\mathrm{e}^{-\lambda(T-t)}-1+\lambda(T-t)\right)\right. \\
&\quad - \frac{\lambda}{2\Sigma^2 \sinh(\lambda(T-t))}\left(x^2 \cosh(\lambda(T-t))\right. \\
&\quad + 2\left(\mathrm{e}^{\lambda(T-t)}-1\right)(Cx+C^2)\Big) \\
&\quad + \left.\frac{\lambda}{2\Sigma^2}x^2\right) \tag{A.3}
\end{aligned}$$

where

$$C = \frac{\Sigma^2}{2\lambda^2}\left(\mathrm{e}^{-\lambda(T-t)}-1\right). \tag{A.4}$$

Likewise one obtains an expression for the denominator:

$$\begin{aligned}
Y = &\sqrt{\frac{\lambda}{2\pi\Sigma^2 \sinh(\lambda(T-t))}} \\
&\times \exp\left(\frac{\lambda}{2\Sigma^2}x^2 - \frac{\lambda}{2\Sigma^2 \sinh(\lambda(T-t))}x^2 \cosh(\lambda(T-t))\right). \tag{A.5}
\end{aligned}$$

Calculating X/Y gives the result

$$I = \exp\left(\frac{\Sigma^2}{2\lambda^2}(T-t) - \frac{1}{\lambda}\tanh\left(\frac{\lambda(T-t)}{2}\right)\left(x+\frac{\Sigma^2}{\lambda^2}\right)\right) \tag{A.6}$$

which leads to equation (33).

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